

STABLE PAIR INVARIANTS UNDER BLOW-UPS

HUA-ZHONG KE

ABSTRACT. We use degeneration formula to study the change of stable pair invariants of 3-folds under blow-ups and obtain some closed blow-up formulae. Related results on Donaldson-Thomas invariants are also discussed. Our results give positive evidence for GW/DT/P correspondence, and also give partial correspondence for varieties not necessarily toric or complete intersections.

Key words: Stable pair invariant, Blow-up, Degeneration formula, Virtual localization, Degenerate contribution, GW/DT/P correspondence

CONTENTS

1. Introduction	1
2. Preliminaries	4
3. Formulae for blow-up at a point	6
4. Formulae for blow-up along a curve	12
References	18

1. INTRODUCTION

Curve counting theories have played prominent roles in both mathematics and physics in the last two decades. For any nonsingular 3-fold X , there are (at least) three different curve counting theories on X . Much studied Gromov-Witten theory counts stable maps from curves to X . Donaldson-Thomas theory [DT, Th] counts one dimensional subschemes in X . Stable pair theory, introduced by Pandharipande and Thomas in [PT], counts pairs (C, D) where $C \subset X$ is an embedded curve and D is a divisor on C . It is conjectured that these three curve counting theories on X are equivalent [MNO1, MNOP2, PT, PP4]. GW/DT/P correspondence has been proved in many important cases, including quintic 3-folds [Br, MOOP, OP, PP4, PP5, T1]. This suggests that many phenomena in one theory have counterparts in the other two theories.

A fundamental problem in Gromov-Witten theory is to understand how Gromov-Witten invariants change under surgeries [LR, R]. For 3-folds, the first breakthrough in this direction is the work of Li and Ruan [LR] on the transformation of Gromov-Witten invariants under flops and extremal small transitions. In birational geometry, blow-up is an elementary surgery, but it is rare to be able to

obtain closed blow-up formulae for Gromov-Witten invariants. In the last twenty years, only a few limited cases were known [Ga, H1, H2, HHKQ, HLR]. It is also important to study the effect of surgeries on Donaldson-Thomas theory. Hu and Li [HL] have studied the transformation of Donaldson-Thomas invariants under blow-ups at points, ordinary flops and extremal small transitions. For general flops between Calabi-Yau 3-folds, Toda [T2] has established the flop formula for Donaldson-Thomas invariants, and hence for stable pair invariants due to the DT/P correspondence in the Calabi-Yau case [Br, T1]. In this paper, we study the transformation of stable pair invariants under blow-ups.

Throughout this paper, let X be an irreducible, nonsingular, projective 3-fold over \mathbb{C} , and $p : \tilde{X} \rightarrow X$ the blow-up of X at a point P or along an irreducible, nonsingular embedded curve C of X . Let E be the exceptional divisor of the blow-up, and $e \in H_2(\tilde{X}, \mathbb{Z})$ the class of a line in the fiber of E . Note that p induces a natural injection via 'pull-back' of 2-cycles

$$p^! = PD_{\tilde{X}} \circ p^* \circ PD_X : H_2(X, \mathbb{Z}) \rightarrow H_2(\tilde{X}, \mathbb{Z}),$$

where the image of $p^!$ is the subset of $H_2(\tilde{X}, \mathbb{Z})$ consisting of 2-cycles having intersection number zero with E . We will compare partition functions Z_P of stable pair invariants of X and those of \tilde{X} , the definition of which will be reviewed in Section 2.

We first consider blow-up at a point.

Theorem 1.1. *Let $p : \tilde{X} \rightarrow X$ be the blow-up at a point. Suppose that $\gamma_1, \dots, \gamma_m \in H^{>0}(X, \mathbb{Q})$, and $d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}$. Then for any $\beta \in H_2(X, \mathbb{Z})$ and $k \in \mathbb{Z}_{>0}$, we have*

$$Z_P(\tilde{X}; q | \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i))_{p^! \beta + k e} = 0.$$

Theorem 1.2. *Let $p : \tilde{X} \rightarrow X$ be the blow-up at a point. Suppose that $\gamma_1, \dots, \gamma_m \in H^{>0}(X, \mathbb{Q})$, and $d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}$. Then for any nonzero $\beta \in H_2(X, \mathbb{Z})$, we have*

$$Z_P(X; q | \prod_{i=1}^m \tau_{d_i}(\gamma_i))_{\beta} = Z_P(\tilde{X}; q | \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i))_{p^! \beta}.$$

Theorem 1.3. *Under the same assumptions as in Theorem 1.2, we have*

$$Z_P(X; q | \tau_0([pt]) \prod_{i=1}^m \tau_{d_i}(\gamma_i))_{\beta} = (1 + q)^2 \cdot Z_P(\tilde{X}; q | \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i))_{p^! \beta - e}.$$

Theorem 1.4. *Under the same assumptions as in Theorem 1.2, we have*

$$Z_P(X; q | \tau_1([pt]) \prod_{i=1}^m \tau_{d_i}(\gamma_i))_{\beta} = \frac{1}{2}(1 - q^2) \cdot Z_P(\tilde{X}; q | \tau_0(-E^2) \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i))_{p^! \beta - e}.$$

We also consider blow-up along a curve.

Theorem 1.5. *Let $p : \tilde{X} \rightarrow X$ be the blow-up along an irreducible, nonsingular embedded curve C with $\int_C c_1(X) \geq 0$. Suppose that $\gamma_1, \dots, \gamma_m \in H^*(X, \mathbb{Q})$ have*

supports away from C , and $d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}$. Then for $\beta \in H_2(X, \mathbb{Z})$ and $k \in \mathbb{Z}_{>0}$, we have

$$Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta + k e} = 0.$$

Theorem 1.6. *Let $p : \tilde{X} \rightarrow X$ be the blow-up along an irreducible, nonsingular embedded curve C with $\int_C c_1(X) > 0$. Suppose that $\gamma_1, \dots, \gamma_m \in H^*(X, \mathbb{Q})$ have supports away from C , and $d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}$. Then for nonzero $\beta \in H_2(X, \mathbb{Z})$, we have*

$$Z_P\left(X; q \middle| \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_{\beta} = Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta}.$$

Theorem 1.7. *Let $p : \tilde{X} \rightarrow X$ be the blow-up along an irreducible, nonsingular embedded curve C with $\int_C c_1(X) > 1$. Suppose that $\gamma_1, \dots, \gamma_m \in H^*(X, \mathbb{Q})$ have supports away from C , and $d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}$. Then for nonzero $\beta \in H_2(X, \mathbb{Z})$, we have*

$$Z_P\left(X; q \middle| \tau_0([C]) \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_{\beta} = (1 + q) \cdot Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta - e}.$$

Remark 1.8. *If $\deg \gamma_i > 2$, then γ_i has support away from C .*

The key tool used in this paper is the degeneration formula [IP, Li, LW, LR, MPT]. Degeneration formula is powerful in the study of structures of Gromov-Witten, Donaldson-Thomas and stable pair theories [HLR, LHH, MOOP, OP, PP1, PP2, PP3, PP4, PP5]. In this paper, the blow-ups of X can be described in terms of semi-stable degenerations of X , and we use degeneration formula to express invariants of X and \tilde{X} in terms of relative invariants of (\tilde{X}, E) . Then we use virtual localization [GP] and degenerate contribution computation [PT] to compute the relevant coefficients to obtain our results.

In [HHKQ], W. He, J. Hu, X. Qi and the author have obtained several blow-up formulae for all genera Gromov-Witten invariants for symplectic manifolds in real dimension six. Assuming GW/P correspondence, many of the results of this paper can be derived from those of [HHKQ]. Moreover, the corresponding results also hold in Donaldson-Thomas theory (except Theorem 1.7). The reason behind the similarity of blow-up formulae for Gromov-Witten, Donaldson-Thomas and stable pair invariants is that the behavior of these invariants under degeneration is similar. Our blow-up formulae give positive evidence for GW/DT/P correspondence. Also, based on known results, our blow-up formulae give partial GW/DT/P correspondence for projective 3-folds not necessarily toric or complete intersections in products of projective spaces.

The author is not able to prove the corresponding result of Theorem 1.7 in Donaldson-Thomas theory. This is because in Donaldson-Thomas theory, free points are allowed to move in the whole variety, which makes the degeneration contribution computation difficult.

In [HL], J. Hu and W.-P. Li have studied the change of Donaldson-Thomas invariants under ordinary flops and extremal small transitions via degeneration formula. We can also study the change of stable pair invariants under these surgeries, using exactly the same arguments as in [HL] to obtain similar results.

The rest of the paper is arranged as follows. In Section 2, we briefly review basic materials of absolute/relative stable pair invariants and the degeneration formula. In Section 3, we consider the case of blow-up at a point. In Section 4, we consider the case of blow-up along a curve.

2. PRELIMINARIES

In this section, we briefly review absolute/relative stable pair invariants and the degeneration formula and fix notations throughout. We refer readers to [LW, MPT, PT] for details.

A stable pair (F, s) on X consists of a pure sheaf F on X supported on a (possibly disconnected) Cohen-Macaulay curve and a section $s \in H^0(X, F)$ with zero dimensional cokernel. For $n \in \mathbb{Z}$ and nonzero $\beta \in H_2(X, \mathbb{Z})$, let $P_n(X, \beta)$ be the moduli space of stable pairs (F, s) with $\chi(F) = n$ and $[F] = \beta$. From the deformation theory of complexes in the derived category, the moduli space $P_n(X, \beta)$ carries a virtual fundamental class.

For $d \in \mathbb{Z}_{\geq 0}$ and $\gamma \in H^*(X, \mathbb{Z})$, the descendant insertion $\tau_d(\gamma)$ is defined as follows. Let

$$\begin{aligned} \pi_X : X \times P_n(X, \beta) &\rightarrow X, \\ \pi_P : X \times P_n(X, \beta) &\rightarrow P_n(X, \beta) \end{aligned}$$

be tautological projections. Let \mathbb{F} be the universal sheaf over $X \times P_n(X, \beta)$. The operation

$$\pi_{P*} \left(\pi_X^*(\gamma) \cdot \text{ch}_{2+d}(\mathbb{F}) \cap \pi_P^*(\cdot) \right) : H_*(P_n(X, \beta), \mathbb{Z}) \rightarrow H_*(P_n(X, \beta), \mathbb{Z})$$

is the action of $\tau_d(\gamma)$. The stable pair invariants with descendant insertions are defined as the virtual integration

$$\left\langle \prod_{i=1}^m \tau_{d_i}(\gamma_i) \right\rangle_{n, \beta} = \int_{P_n(X, \beta)} \prod_{i=1}^m \tau_{d_i}(\gamma_i) \left([P_n(X, \beta)]^{\text{vir}} \right),$$

where $d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}$, and $\gamma_1, \dots, \gamma_m \in H^*(X, \mathbb{Z})$. Denote the partition function of stable pair invariants as

$$Z_P \left(X; q \middle| \prod_{i=1}^m \tau_{d_i}(\gamma_i) \right)_{\beta} = \sum_{n \in \mathbb{Z}} \left\langle \prod_{i=1}^m \tau_{d_i}(\gamma_i) \right\rangle_{n, \beta} q^n.$$

Let $S \subset X$ be a nonsingular divisor. For $n \in \mathbb{Z}$ and nonzero $\beta \in H_2(X, \mathbb{Z})$ with $\int_{\beta} [S] \geq 0$, let $P_n(X/S, \beta)$ be the moduli space of relative stable pairs, which carries a virtual fundamental class of degree $\int_{\beta} c_1(X)$. We have the following natural

morphism

$$\epsilon : P_n(X/S, \beta) \rightarrow \text{Hilb}(S, \int_{\beta} [S])$$

The pull-back of cohomology classes of $\text{Hilb}(S, \int_{\beta} [S])$ gives relative insertions.

Let us briefly recall Nakajima basis for the cohomology of Hilbert schemes of points of S . Let $\{\delta_i\}$ be a basis of $H^*(S, \mathbb{Q})$ with dual basis $\{\delta^i\}$. For any cohomology weighted partition η with respect to the basis $\{\delta_i\}$, Nakajima constructed a cohomology class $C_{\eta} \in H^*(\text{Hilb}(S, |\eta|), \mathbb{Q})$. The Nakajima basis of $H^*(\text{Hilb}(S, d), \mathbb{Q})$ is the set $\{C_{\eta}\}_{|\eta|=d}$. We refer readers to [Na] for more details.

The partition function of relative stable pair invariants are defined by

$$Z_P(X/S; q | \prod_{i=1}^m \tau_{d_i}(\gamma_i) | \eta)_{\beta} = \sum_{n \in \mathbb{Z}} q^n \int_{[P_n(X/S, \beta)]^{\text{vir}}} \prod_{i=1}^m \tau_{d_i}(\gamma_i) \cdot \epsilon^* C_{\eta}.$$

Let $\pi : \chi \rightarrow \mathbb{A}^1$ be a nonsingular 4-fold over \mathbb{A}^1 such that $\chi_t = \pi^{-1}(t) \cong X$ for $t \neq 0$ and χ_0 is a union of two irreducible nonsingular projective 3-folds X_1 and X_2 intersecting transversally along a nonsingular projective surface S .

Consider the natural inclusion maps

$$i_t : X = \chi_t \longrightarrow \chi, \quad i_0 : \chi_0 \longrightarrow \chi,$$

and the gluing map

$$g = (j_1, j_2) : X_1 \coprod X_2 \longrightarrow \chi_0.$$

We have

$$H_2(X, \mathbb{Z}) \xrightarrow{i_{t*}} H_2(\chi, \mathbb{Z}) \xleftarrow{i_{0*}} H_2(\chi_0, \mathbb{Z}) \xleftarrow{g^*} H_2(X_1, \mathbb{Z}) \oplus H_2(X_2, \mathbb{Z}),$$

where i_{0*} is an isomorphism since there exists a deformation retract from χ to χ_0 (see [CI]). Also, since the family $\chi \rightarrow \mathbb{A}^1$ comes from a trivial family, it follows that each $\gamma \in H^*(X, \mathbb{Q})$ has global liftings such that the restriction $\gamma(t)$ on χ_t is defined for all t .

The degeneration formula for stable pair theory expresses absolute invariants of X via relative invariants of (X_1, S) and (X_2, S) :

$$\begin{aligned} & Z_P(X; q | \prod_{i=1}^m \tau_{d_i}(\gamma_i) | \eta)_{\beta} \\ &= \sum Z_P(X_1/S; q | \prod_{i \in P_1} \tau_{d_i}(J_1^* \gamma_i(0)) | \eta)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} \mathfrak{z}(\eta)}{q^{|\eta|}} \cdot Z_P(X_2/S; q | \prod_{i \in P_2} \tau_{d_i}(J_2^* \gamma_i(0)) | \eta^{\vee})_{\beta_2}, \end{aligned}$$

where $\mathfrak{z}(\eta) = |\text{Aut}(\eta)| \cdot \prod_{i=1}^{\ell(\eta)} \eta_i$, η^{\vee} is defined by taking the Poincaré duals of the cohomology weights of η , and the sum is over cohomology weighted partitions η , degree splittings $i_{t*} \beta = i_{0*}(j_{1*} \beta_1 + j_{2*} \beta_2)$, and marking partitions $P_1 \coprod P_2 = \{1, \dots, m\}$. In particular, if (η, β_1, β_2) has nontrivial contribution in the degeneration formula, then we have the following dimension constraint:

$$\text{vdim}_{\mathbb{C}} P_n(X_1/S, \beta_1) + \text{vdim}_{\mathbb{C}} P_n(X_2/S, \beta_2) = \text{vdim}_{\mathbb{C}} P_n(X, \beta) + 2|\eta|.$$

3. FORMULAE FOR BLOW-UP AT A POINT

In this section, we consider blow-up at a point and prove Theorem 1.1, 1.2, 1.3, and 1.4. We always assume that total degrees of insertions match the virtual dimensions of the moduli spaces, since otherwise the required equalities are trivial.

Throughout this section, we let H be the hyperplane at infinity in \mathbb{P}^3 , and $\tilde{\mathbb{P}}^3$ is the blow-up of \mathbb{P}^3 at a point not in H .

We first prove Theorem 1.1. Degenerate \tilde{X} along E , and by the degeneration formula, we have

$$(1) \quad \begin{aligned} & Z_P(\tilde{X}; q | \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i))_{p^! \beta + k e} \\ &= \sum Z_P(\tilde{\mathbb{P}}^3/H; q | \eta)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} 3(\eta)}{q^{|\eta|}} \cdot Z_P(\tilde{X}/E; q | \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i) | \eta^\vee)_{\beta_2}, \end{aligned}$$

where we have assumed that the class $p^* \gamma_i$ has support away from E . By our assumption that degrees match the virtual dimensions, we have

$$\text{vdim}_{\mathbb{C}} P_n(\tilde{X}, p^! \beta) = \frac{1}{2} \sum_{i=1}^m \gamma_i + \sum_{i=1}^m d_i - m.$$

Suppose that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nonzero contribution in (1). Then

$$\begin{aligned} \text{vdim}_{\mathbb{C}} P_n(\tilde{\mathbb{P}}^3/H, \beta_1) &= \int_{\beta_1} c_1(\tilde{\mathbb{P}}^3), \\ \text{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\tilde{\mathbb{P}}^3) - |\eta| = \ell(\eta).$$

Let $L \in H_2(\tilde{\mathbb{P}}^3, \mathbb{Z})$ be the class of the total transform of a line in \mathbb{P}^3 . Then we have the following natural decomposition

$$H_2(\tilde{\mathbb{P}}^3, \mathbb{Z}) = \mathbb{Z}F \oplus \mathbb{Z}L.$$

We have the following constraint for β_1 :

$$\begin{cases} \beta_1 \cdot H &= |\eta|, \\ \beta_1 \cdot E &= -k. \end{cases}$$

So we have $\int_{\beta_1} c_1(\tilde{\mathbb{P}}^3) = 4|\eta| + 2k$. Now the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + 3|\eta| + 2k = \ell(\eta).$$

We observe that no partition satisfies the dimension constraint, and this proves Theorem 1.1.

Next, we prove Theorem 1.2. We divide the proof of Theorem 1.2 into two comparison lemmas of stable pair invariants.

Lemma 3.1. *Under the assumptions as in Theorem 1.2, we have*

$$Z_P\left(X; q \middle| \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta = Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta}.$$

Proof. Degenerate X at a point P , and by the degeneration formula, we have

$$\begin{aligned} (2) \quad & Z_P\left(X; q \middle| \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta \\ &= \sum Z_P\left(\mathbb{P}^3/H; q \middle| \eta\right)_{\beta_1} \cdot \frac{(-1)^{|\eta|-\ell(\eta)} 3(\eta)}{q^{|\eta|}} \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i) \middle| \eta^\vee\right)_{\beta_2}, \end{aligned}$$

where we have assumed that the support of γ_i is away from P . By our assumption that total degrees of insertions match the virtual dimensions of moduli spaces, we have

$$\text{vdim}_{\mathbb{C}} P_n(X, \beta) = \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i - m.$$

Suppose that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nonzero contribution in (2). Then

$$\begin{aligned} \text{vdim}_{\mathbb{C}} P_n(\mathbb{P}^3/H, \beta_1) &= \int_{\beta_1} c_1(\mathbb{P}^3), \\ \text{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\mathbb{P}^3) - |\eta| = \ell(\eta).$$

Note that $\beta_1 \cdot H = |\eta|$, and hence $\beta_1 = |\eta|L$, which implies that

$$\int_{\beta_1} c_1(\mathbb{P}^3) = 4|\eta|.$$

Now the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + 3|\eta| = \ell(\eta).$$

So the dimension constraint holds only if $\eta = \emptyset$, which implies Lemma 3.1. \square

Lemma 3.2. *Under the assumptions as in Theorem 1.2, we have*

$$Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta} = Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta}.$$

Proof. Degenerate \tilde{X} along E , and by the degeneration formula, we have

$$(3) \quad \begin{aligned} & Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta} \\ &= \sum Z_P(\tilde{\mathbb{P}}^3/H; q \middle| \eta)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} \mathfrak{z}(\eta)}{q^{|\eta|}} \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i) \middle| \eta^\vee\right)_{\beta_2}, \end{aligned}$$

where we have assumed that the class $p^* \gamma_i$ has support away from E . By our assumption that degrees match the virtual dimensions, we have

$$\mathrm{vdim}_{\mathbb{C}} P_n(\tilde{X}, p^! \beta + ke) = \frac{1}{2} \sum_{i=1}^m \gamma_i + \sum_{i=1}^m d_i - m.$$

Suppose that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nonzero contribution in (3). Then

$$\begin{aligned} \mathrm{vdim}_{\mathbb{C}} P_n(\tilde{\mathbb{P}}^3/H, \beta_1) &= \int_{\beta_1} c_1(\tilde{\mathbb{P}}^3), \\ \mathrm{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\tilde{\mathbb{P}}^3) - |\eta| = \ell(\eta).$$

Let $L \in H_2(\tilde{\mathbb{P}}^3, \mathbb{Z})$ be the class of the total transform of a line in \mathbb{P}^3 . Then we have the following natural decomposition

$$H_2(\tilde{\mathbb{P}}^3, \mathbb{Z}) = \mathbb{Z}F \oplus \mathbb{Z}L.$$

We have the following constraint for β_1 :

$$\begin{cases} \beta_1 \cdot H &= |\eta|, \\ \beta_1 \cdot E &= 0. \end{cases}$$

So we have $\beta_1 = |\eta|L$, and hence $\int_{\beta_1} c_1(\tilde{\mathbb{P}}^3) = 4|\eta|$. Now the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + 3|\eta| = \ell(\eta).$$

So the dimension constraint holds only if $\eta = \emptyset$, which implies Lemma 3.2. \square

The above comparison results give Theorem 1.2.

To prove Theorem 1.3, we need the following two comparison lemmas.

Lemma 3.3. *Under the same assumptions as in Theorem 1.3, we have*

$$\begin{aligned} & Z_P\left(X; q|\tau_0([pt]) \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta \\ &= Z_P\left(\mathbb{P}^3/H; q|\tau_0([pt])|(1, [pt])\right)_L \cdot \frac{1}{q} \cdot Z_P\left(\tilde{X}/E; q|\prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)|(1, \mathbb{1})\right)_{p^! \beta - e}, \end{aligned}$$

where L is the class of a line.

Proof. Degenerate X at a point P , and by the degeneration formula, we have

$$\begin{aligned} (4) \quad & Z_P\left(X; q|\tau_0([pt]) \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta \\ &= \sum Z_P\left(\mathbb{P}^3/H; q|\tau_0([pt])|\eta\right)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} \mathfrak{z}(\eta)}{q^{|\eta|}} \cdot Z_P\left(\tilde{X}/E; q|\prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)|\eta^\vee\right)_{\beta_2}, \end{aligned}$$

where we have assumed that the support of γ_i is away from P . By our assumption that total degrees of insertions match the virtual dimensions of moduli spaces, we have

$$\text{vdim}_{\mathbb{C}} P_n(X, \beta) = \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + 2 - m.$$

Suppose that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nonzero contribution in (4). Then

$$\begin{aligned} \text{vdim}_{\mathbb{C}} P_n(\mathbb{P}^3/H, \beta_1) &= \int_{\beta_1} c_1(\mathbb{P}^3), \\ \text{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\mathbb{P}^3) - |\eta| = 2 + \ell(\eta).$$

Note that $\beta_1 \cdot H = |\eta|$, and hence $\beta_1 = |\eta|L$, which implies that

$$\int_{\beta_1} c_1(\mathbb{P}^3) = 4|\eta|.$$

Now the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + 3|\eta| = 2 + \ell(\eta).$$

So the dimension constraint holds only if $\eta = (1, [pt])$, which implies Lemma 3.3. \square

Lemma 3.4.

$$\begin{aligned} & Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta - e} \\ &= Z_P\left(\tilde{\mathbb{P}}^3/H; q \middle| (1, [pt])\right)_F \cdot \frac{1}{q} \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i) \middle| (1, \mathbb{1})\right)_{p^! \beta - e}, \end{aligned}$$

where F is the fiber class of $\tilde{\mathbb{P}}^3 \cong \mathbb{P}_H(\mathcal{O}(1) \oplus \mathcal{O})$.

Proof. Degenerate \tilde{X} along E , and by the degeneration formula, we have

$$\begin{aligned} (5) \quad & Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta - e} \\ &= \sum Z_P(\tilde{\mathbb{P}}^3/H; q \middle| \eta)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} \mathfrak{z}(\eta)}{q^{|\eta|}} \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i) \middle| \eta^\vee\right)_{\beta_2}, \end{aligned}$$

where we have assumed that the class $p^* \gamma_i$ has support away from E . By our assumption that degrees match the virtual dimensions, we have

$$\text{vdim}_{\mathbb{C}} P_n(\tilde{X}, p^! \beta - e) = \frac{1}{2} \sum_{i=1}^m \gamma_i + \sum_{i=1}^m d_i - m.$$

Suppose that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nonzero contribution in (5). Then

$$\begin{aligned} \text{vdim}_{\mathbb{C}} P_n(\tilde{\mathbb{P}}^3/H, \beta_1) &= \int_{\beta_1} c_1(\tilde{\mathbb{P}}^3), \\ \text{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\tilde{\mathbb{P}}^3) - |\eta| = \ell(\eta).$$

Let $L \in H_2(\tilde{\mathbb{P}}^3, \mathbb{Z})$ be the class of the total transform of a line in \mathbb{P}^3 . Then we have the following natural decomposition

$$H_2(\tilde{\mathbb{P}}^3, \mathbb{Z}) = \mathbb{Z}F \oplus \mathbb{Z}L.$$

We have the following constraint for β_1 :

$$\begin{cases} \beta_1 \cdot H &= |\eta|, \\ \beta_1 \cdot E &= 1. \end{cases}$$

So we have $\beta_1 = F + (|\eta| - 1)L$, and hence $\int_{\beta_1} c_1(\tilde{\mathbb{P}}^3) = 4|\eta| - 2$. Now the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + 3|\eta| = 2 + \ell(\eta).$$

So the dimension constraint holds only if $\eta = (1, [pt])$, which implies Lemma 3.4. \square

Proof of Theorem 1.3: By Lemma 3.3 and 3.4, in the particular case $X = \mathbb{P}^3$, we have

$$\frac{Z_P\left(\mathbb{P}^3; q|\tau_0([pt])^2\right)_L}{Z_P\left(\tilde{\mathbb{P}}^3; q|\tau_0([pt])\right)_F} = \frac{Z_P\left(\mathbb{P}^3/H; q|\tau_0([pt])(1, [pt])\right)_L}{Z_P\left(\tilde{\mathbb{P}}^3/H; q|(1, [pt])\right)_F}.$$

Now by virtual localization [GP] or by (4.2) in [PT], we have

$$\begin{aligned} Z_P\left(\mathbb{P}^3; q|\tau_0([pt])^2\right)_L &= q(1+q)^2, \\ Z_P\left(\tilde{\mathbb{P}}^3; q|\tau_0([pt])\right)_F &= q, \end{aligned}$$

which gives Theorem 1.3.

Theorem 1.4 relies on the following Lemma 3.5 and 3.6, the proof of which is analogous to that of Lemma 3.3, 3.4 respectively.

Lemma 3.5.

$$\begin{aligned} & Z_P\left(X; q|\tau_1([pt]) \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta \\ &= Z_P\left(\mathbb{P}^3/H; q|\tau_1([pt])(1, [L])\right)_L \cdot \frac{1}{q} \cdot Z_P\left(\tilde{X}/E; q|\prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)(1, [L])\right)_{p'\beta-e}. \end{aligned}$$

Lemma 3.6.

$$\begin{aligned} & Z_P\left(\tilde{X}; q|\tau_0(-E^2) \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p'\beta-e} \\ &= Z_P\left(\tilde{X}/H; q|\tau_0(-E^2)(1, [L])\right)_F \cdot \frac{1}{q} \cdot Z_P\left(\tilde{X}/E; q|\prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)(1, [L])\right)_{p'\beta-e}. \end{aligned}$$

Proof of Theorem 1.4: By Lemma 3.5 and 3.6, in the particular case $X = \mathbb{P}^3$, we have

$$\frac{Z_P\left(\mathbb{P}^3; q|\tau_1([pt])\tau_0([L])\right)_L}{Z_P\left(\mathbb{P}^3; q|\tau_0(-E^2)\tau_0([L])\right)_F} = \frac{Z_P\left(\mathbb{P}^3/H; q|\tau_1([pt])(1, [L])\right)_L}{Z_P\left(\tilde{\mathbb{P}}^3/H; q|\tau_0(-E^2)(1, [L])\right)_F},$$

By virtual localization [GP], we have

$$\begin{aligned} Z_P\left(\mathbb{P}^3; q|\tau_1([pt])\tau_0([L])\right)_L &= \frac{1}{2}q(1-q^2), \\ Z_P\left(\mathbb{P}^3; q|\tau_0(-E^2)\tau_0([L])\right)_F &= q, \end{aligned}$$

which gives Theorem 1.4.

4. FORMULAE FOR BLOW-UP ALONG A CURVE

In this section, we consider blow-up along a nonsingular embedded curve and prove Theorem 1.5, 1.6 and 1.7. We always assume that total degrees of insertions match the virtual dimensions of the moduli spaces, since otherwise the required equalities are trivial.

Throughout this section, we let N_C be the normal bundle of C in X , and N_E the normal bundle of the exceptional divisor E in \tilde{X} .

We first prove Theorem 1.5. Degenerate \tilde{X} along E , and by the degeneration formula, we have

$$\begin{aligned}
 (6) \quad & Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta + ke} \\
 &= \sum Z_P\left(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty; q \middle| \eta\right)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} \mathfrak{z}(\eta)}{q^{|\eta|}} \\
 &\quad \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i) |\eta^\vee\right)_{\beta_2},
 \end{aligned}$$

where we have assumed that the support of $p^* \gamma_i$ is away from E , and $D_\infty = \mathbb{P}_E(N_E \oplus \{0\})$. Recall that we have assumed that

$$\text{vdim}_{\mathbb{C}} P_n(\tilde{X}, p^! \beta + ke) = \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i - m.$$

Assume that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nontrivial contribution in (6), and then

$$\begin{aligned}
 \text{vdim}_{\mathbb{C}} P_n(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty, \beta_1) &= \int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)), \\
 \text{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m.
 \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)) - |\eta| = \ell(\eta).$$

Let ξ be the tautological line bundle of $\mathbb{P}_E(N_E \oplus \mathcal{O}_E)$. Then Euler exact sequence gives

$$c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)) = \pi^* c_1(E) + \pi^* c_1(N_E) - 2c_1(\xi),$$

where $\pi : \mathbb{P}_E(N_E \oplus \mathcal{O}_E) \rightarrow E$ is the natural projection. Note that N_E is the tautological line bundle of $E \cong \mathbb{P}_C(N_C)$, and so

$$c_1(E) = \pi_E^* c_1(X)|_C - 2c_1(N_E),$$

where $\pi_E : E \rightarrow C$ is the natural projection. Therefore,

$$c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)) = (\pi_E \circ \pi)^* c_1(X)|_C - \pi^* c_1(N_E) - 2c_1(\xi).$$

Note that we have the following natural decomposition

$$H_2(\mathbb{P}_E(N_E \oplus \mathcal{O}_E), \mathbb{Z}) \cong \mathbb{Z}F \oplus H_2(E, \mathbb{Z}),$$

and we can write

$$\beta_1 = aF + \pi_*\beta_1, \text{ for some } a \in \mathbb{Z}_{\geq 0}.$$

We have the following constraints for β_1 :

$$\begin{cases} \beta_1 \cdot D_\infty &= |\eta|, \\ \beta_1 \cdot E &= -k, \end{cases}$$

and this gives

$$\pi_*\beta_1 \cdot E = -|\eta| - k.$$

Note that $-c_1(\xi)$ is the Poincaré dual of the divisor D_∞ in $\mathbb{P}_E(N_E \oplus \mathcal{O}_E)$, and therefore

$$\int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)) = \int_{(\pi_E \circ \pi)_*\beta_1} c_1(X)|_C + 3|\eta| + k.$$

Hence the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{(\pi_E \circ \pi)_*\beta_1} c_1(X)|_C + 2|\eta| + k = \ell(\eta).$$

We observe that no partition satisfies the dimension constraint, which gives Theorem 1.5.

Next, we prove Theorem 1.6. We divide the proof into two comparison lemmas of stable pair invariants.

Lemma 4.1. *Under the same assumptions as in Theorem 1.6, we have*

$$Z_P\left(X; q \middle| \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta = Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^*\gamma_i)\right)_{p^*\beta}.$$

Proof. Degenerate X along C , and by the degeneration formula, we have:

$$\begin{aligned} (7) \quad & Z_P\left(X; q \middle| \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta \\ &= \sum Z_P\left(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)/D_\infty; q \middle| \eta\right)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} 3(\eta)}{q^{|\eta|}} \\ & \quad \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^*\gamma_i)|\eta^\vee\right)_{\beta_2}, \end{aligned}$$

where we have assumed that the support of γ_i is away from C , and $D_\infty = \mathbb{P}_C(N_C \oplus \{0\})$. Recall that we have assumed that

$$\text{vdim}_{\mathbb{C}} P_n(X, \beta) = \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i - m.$$

Suppose that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nonzero contribution in (7). Then

$$\begin{aligned} \text{vdim}_{\mathbb{C}} P_n(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)/D_\infty, \beta_1) &= \int_{\beta_1} c_1(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)), \\ \text{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)) - |\eta| = \ell(\eta).$$

Let ξ be the tautological line bundle of $\mathbb{P}_C(N_C \oplus \mathcal{O}_C)$, and we have

$$c_1(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)) = \pi^* c_1(X)|_C - 3c_1(\xi),$$

where $\pi : \mathbb{P}_C(N_C \oplus \mathcal{O}_C) \rightarrow C$ is the natural projection. Note that $-c_1(\xi)$ is the Poincaré dual of the divisor D_∞ in $\mathbb{P}_C(N_C \oplus \mathcal{O}_C)$. Since $|\eta| = \beta_1 \cdot D_\infty$, it follows that

$$\int_{\beta_1} c_1(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)) = \int_{\pi_* \beta_1} c_1(X)|_C + 3|\eta|.$$

Therefore, dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\pi_* \beta_1} c_1(X)|_C + 2|\eta| = \ell(\eta).$$

The dimension constraint holds only if

$$\eta = \emptyset, \quad \int_{\pi_* \beta_1} c_1(X)|_C = 0,$$

which implies Lemma 4.1. □

Lemma 4.2. *Under the same assumptions as in Theorem 1.6, we have*

$$Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta} = Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta}.$$

Proof. Degenerate \tilde{X} along E , and by the degeneration formula, we have

$$\begin{aligned} (8) \quad & Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta} \\ &= \sum Z_P\left(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty; q \middle| \eta\right)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} \mathfrak{z}(\eta)}{q^{|\eta|}} \\ & \quad \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i) |\eta^\vee\right)_{\beta_2}, \end{aligned}$$

where we have assumed that the support of $p^*\alpha_i$ is away from E , and $D_\infty = \mathbb{P}_E(N_E \oplus \{0\})$. Recall that we have assumed that

$$\mathrm{vdim}_{\mathbb{C}} P_n(\tilde{X}, p^!\beta) = \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i - m.$$

Assume that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nontrivial contribution in (8), and then

$$\begin{aligned} \mathrm{vdim}_{\mathbb{C}} P_n(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty, \beta_1) &= \int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)), \\ \mathrm{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)) - |\eta| = \ell(\eta).$$

Following the notations in the proof of Theorem 1.5, we have the following constraints for β_1 :

$$\begin{cases} \beta_1 \cdot D_\infty &= |\eta|, \\ \beta_1 \cdot E &= 0, \end{cases}$$

and this gives

$$\pi_* \beta_1 \cdot E = -|\eta|,$$

which implies that

$$\int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)) = \int_{(\pi_E \circ \pi)_* \beta_1} c_1(X)|_C + 3|\eta|.$$

Hence the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{(\pi_E \circ \pi)_* \beta_1} c_1(X)|_C + 2|\eta| = \ell(\eta).$$

So the dimension constraint holds only if

$$\eta = \emptyset, \quad \int_{(\pi_E \circ \pi)_* \beta_1} c_1(X)|_C = 0,$$

which implies Lemma 4.2. □

The above two comparison results give Theorem 1.6.

To prove Theorem 1.7, we need the following two lemmas.

Lemma 4.3. *Under the same assumptions as in Theorem 1.7, we have*

$$\begin{aligned} & Z_P\left(X; q|\tau_0([C]) \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta \\ &= Z_P\left(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)/D_\infty; q|\tau_0([C])|(1, [pt])\right)_F \cdot \frac{1}{q} \cdot Z_P\left(\tilde{X}/E; q|\prod_{i=1}^m \tau_{d_i}(p^*\gamma_i)|(1, \mathbb{1})\right)_{p^!\beta-e}, \end{aligned}$$

where $D_\infty = \mathbb{P}_C(N_C \oplus \{0\})$, and F is the class of a line in the fiber of $\mathbb{P}_C(N_C \oplus \mathcal{O}_C)$.

Proof. Degenerate X along C , and by the degeneration formula, we have:

$$\begin{aligned} (9) \quad & Z_P\left(X; q|\tau_0([C]) \prod_{i=1}^m \tau_{d_i}(\gamma_i)\right)_\beta \\ &= \sum Z_P\left(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)/D_\infty; q|\tau_0[C]|\eta\right)_{\beta_1} \cdot \frac{(-1)^{|\eta|-\ell(\eta)} 3(\eta)}{q^{|\eta|}} \\ & \quad \cdot Z_P\left(\tilde{X}/E; q|\prod_{i=1}^m \tau_{d_i}(p^*\gamma_i)|\eta^\vee\right)_{\beta_2}, \end{aligned}$$

where we have assumed that the support of γ_i is away from C . Recall that we have assumed that

$$\text{vdim}_{\mathbb{C}} P_n(X, \beta) = \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + 1 - m.$$

Suppose that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nonzero contribution in (9). Then

$$\begin{aligned} \text{vdim}_{\mathbb{C}} P_n(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)/D_\infty, \beta_1) &= \int_{\beta_1} c_1(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)), \\ \text{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)) - |\eta| = 1 + \ell(\eta).$$

As in the proof of Lemma 4.1, one can check that

$$\int_{\beta_1} c_1(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)) = \int_{\pi_* \beta_1} c_1(X)|_C + 3|\eta|,$$

and the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\pi_* \beta_1} c_1(X)|_C + 2|\eta| = 1 + \ell(\eta).$$

The dimension constraint holds only if

$$\eta = (1, [pt]), \quad \int_{\pi_* \beta_1} c_1(X)|_C = 0,$$

which implies Lemma 4.3. \square

Lemma 4.4. *Under the same assumptions as in Theorem 1.7, we have*

$$\begin{aligned} & Z_P\left(\tilde{X}; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta - e} \\ &= Z_P\left(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty; q \middle| \tau_0(E)(1, [pt])\right)_F \cdot \frac{1}{q} \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)(1, \mathbb{1})\right)_{p^! \beta - e}, \end{aligned}$$

where $D_\infty = \mathbb{P}_E(N_E \oplus \{0\})$, and F the class of a line in the fiber of $\mathbb{P}_E(N_E \oplus \mathcal{O}_E)$.

Proof. Degenerate \tilde{X} along E , and by the degeneration formula, we have

$$\begin{aligned} (10) \quad & Z_P\left(\tilde{X}; q \middle| \tau_0(E) \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i)\right)_{p^! \beta - e} \\ &= \sum Z_P\left(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty; q \middle| \eta\right)_{\beta_1} \cdot \frac{(-1)^{|\eta| - \ell(\eta)} \mathfrak{z}(\eta)}{q^{|\eta|}} \\ & \quad \cdot Z_P\left(\tilde{X}/E; q \middle| \prod_{i=1}^m \tau_{d_i}(p^* \gamma_i) |\eta^\vee\right)_{\beta_2}, \end{aligned}$$

where we have assumed that the support of $p^* \gamma_i$ is away from E . Recall that we have assumed that

$$\text{vdim}_{\mathbb{C}} P_n(\tilde{X}, p^! \beta - e) = \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i - m.$$

Assume that $(\eta = \{(\eta_i, \delta_{j_i})\}_{i=1}^{\ell(\eta)}, \beta_1, \beta_2)$ has nontrivial contribution in (10), and then

$$\begin{aligned} \text{vdim}_{\mathbb{C}} P_n(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty, \beta_1) &= \int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)), \\ \text{vdim}_{\mathbb{C}} P_n(\tilde{X}/E, \beta_2) &= \frac{1}{2} \sum_{i=1}^m \deg \gamma_i + \sum_{i=1}^m d_i + \frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} - \ell(\eta) + |\eta| - m. \end{aligned}$$

So by the dimension constraint,

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)) - |\eta| = \ell(\eta).$$

Following the notations in the proof of Theorem 1.5, we have the following constraints for β_1 :

$$\begin{cases} \beta_1 \cdot D_\infty &= |\eta|, \\ \beta_1 \cdot E &= 1, \end{cases}$$

and this gives

$$\pi_* \beta_1 \cdot E = -|\eta| + 1,$$

which implies that

$$\int_{\beta_1} c_1(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)) = \int_{(\pi_E \circ \pi)_* \beta_1} c_1(X)|_C + 3|\eta| - 1.$$

Hence the dimension constraint becomes

$$\frac{1}{2} \sum_{i=1}^{\ell(\eta)} \deg \delta^{j_i} + \int_{(\pi_E \circ \pi)_* \beta_1} c_1(X)|_C + 2|\eta| = 1 + \ell(\eta).$$

So the dimension constraint holds only if

$$\eta = (1, [pt]), \quad \int_{(\pi_E \circ \pi)_* \beta_1} c_1(X)|_C = 0,$$

which implies Lemma 4.4. \square

Proof of Theorem 1.7: By Lemma 4.3 and 4.4, in the particular case $X = \mathbb{P}_C(N_C \oplus \mathcal{O}_C)$, we have

$$\frac{Z_P\left(\mathbb{P}_C(N_C \oplus \mathcal{O}_C); q|\tau_0([C])\tau_0([pt])\right)_F}{Z_P\left(\mathbb{P}_E(N_E \oplus \mathcal{O}_E); q|\tau_0(E)\tau_0([pt])\right)_F} = \frac{Z_P\left(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)/D_\infty; q|\tau_0([C])(1, [pt])\right)_F}{Z_P\left(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty; q|\tau_0(E)(1, [pt])\right)_F}.$$

By (4.2) in [PT], we have

$$\begin{aligned} Z_P\left(\mathbb{P}_C(N_C \oplus \mathcal{O}_C)/D_\infty; q|\tau_0([C])(1, [pt])\right)_F &= q(1+q), \\ Z_P\left(\mathbb{P}_E(N_E \oplus \mathcal{O}_E)/D_\infty; q|\tau_0(E)(1, [pt])\right)_F &= q. \end{aligned}$$

which gives Theorem 1.7.

Acknowledgements. The author would like to thank Jianxun Hu for valuable comments on earlier versions of this paper, and Yongbin Ruan, Wei-Ping Li and Zhenbo Qin for many helpful discussions.

REFERENCES

- [Br] Bridgeland, T. Hall algebras and curve-counting invariants, J. Amer. Math. Soc. 24 (2011), no. 4, 969-998.
- [Cl] Clemens, C. H., Degeneration of Kähler manifolds, Duke Math. J., 44 (1977), no. 2, 215-290.
- [DT] Donaldson, S., Thomas, R., Gauge theory in higher dimensions, in The Geometric Universe: Science, Geometry, and the Work of Roger Penrose, S. Hugget et. al eds., Oxford Univ. Press, (1998).
- [Ga] Gathmann, A., Gromov-Witten invariants of blow-ups, J. Algebraic Geom. 10 (2001), no. 3, 399-432.
- [GP] Graber, T., Pandharipande, R., Localization of virtual classes, Invent.math. 135, 487-518(1999).
- [H1] Hu, J., Gromov-Witten invariants of blow-ups along points and curves, Math.Z. 233, 709-739(2000).
- [H2] Hu, J., Gromov-Witten invariants of blow-ups along surfaces, Compositio Math. 125 (2001), no. 3, 345-352.
- [HHKQ] He, W., Hu, J., Ke, H., Qi, X., Blow-up formulae of high genus Gromov-Witten invariants in dimension six, arXiv:1402.4221.

- [HL] Hu, J., Li, W.-P., The Donaldson-Thomas invariants under blowups and flops, *J. Differential Geom.* 90 (2012), no. 3, 391-411.
- [HLR] Hu, J., Li, T.-J., Ruan, Y., Birational cobordism invariance of uniruled symplectic manifolds, *Invent. Math.*, 172(2008), 231-275.
- [IP] Ionel, E., Parker, T., The Symplectic Sum Formula for Gromov-Witten Invariants, *Ann. of Math.*, 159(3), 2004, 935-1025.
- [Li] Li, J., A degeneration formula of GW-invariants, *J. Diff. Geom.*, 60(2002),199-293.
- [LHH] Lee, Y.-P., Lin, H.-W., Wang, C.-L., Flops, motives, and invariance of quantum rings, *Ann. of Math.* (2) 172 (2010), no. 1, 243290.
- [LW] Li, J., Wu, B., Good degeneration of quot-schemes and coherent systems, *arXiv:1110.0390v1*.
- [LR] Li, A.-M., Ruan, Y., Symplectic surgery and Gromov-Witten invariants of Calabi 3-folds, *Invent. Math.* 145 (2001), no. 1, 151-218.
- [MNOP1] Maulik, D., Nekrasov, N., Okounkov, A., Pandharipande, R., Gromov-Witten theory and Donaldson-Thomas theory I, *Compos. Math.* 142 (2006), no. 5, 12631285.
- [MNOP2] Maulik, D., Nekrasov, N., Okounkov, A., Pandharipande, R., Gromov-Witten theory and Donaldson-Thomas theory II, *Compos. Math.* 142 (2006), no. 5, 12861304.
- [MOOP] Maulik, D., Oblomkov, A., Okounkov, A., Pandharipande, R., Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds, *Invent. Math.* 186 (2011), no. 2, 435479.
- [MPT] Maulik, D., Pandharipande, R., Thomas, R. P., Curves on K3 surfaces and modular forms. With an appendix by A. Pixton, *J. Topol.* 3 (2010), no. 4, 937996.
- [Na] Nakajima, H., Lectures on Hilbert schemes of points on surfaces, *University Lecture Series*, 18. American Mathematical Society, Providence, RI, 1999.
- [OP] Okounkov, A., Pandharipande, R., The local Donaldson-Thomas theory of curves, *Geom. Topol.* 14 (2010), no. 3, 15031567.
- [PP1] Pandharipande, R., Pixton, A., Descendents on local curves: stationary theory, *Geometry and arithmetic*, 283307, EMS Ser. Congr. Rep., Eur. Math. Soc., Zrich, 2012.
- [PP2] Pandharipande, R., Pixton, A., Descendents on local curves: rationality, *Compos. Math.* 149 (2013), no. 1, 81124.
- [PP3] Pandharipande, R., Pixton, A., Descendent theory for stable pairs on toric 3-folds, *J. Math. Soc. Japan* 65 (2013), no. 4, 13371372.
- [PP4] Pandharipande, R., Pixton, A., Gromov-Witten/Pairs descendance correspondence for toric 3-folds, *arXiv:1203.0468v2*.
- [PP5] Pandharipande, R., Pixton, A., Gromov-Witten/Pairs correspondence for the quintic 3-fold, *arXiv:1206.5490v1*.
- [PT] Pandharipande, R., Thomas, R. P., Curve counting via stable pairs in the derived category, *Invent. Math.* 178 (2009), no. 2, 407447.
- [R] Ruan, Y., Surgery, quantum cohomology and birational geometry, *Northern California Symplectic Geometry Seminar AMS Translations, Series 2*, 1999 (196), 183-198.
- [Th] Thomas, R., A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on K3 fibrations, *J. Differential Geom.* 53 (1999), 367-438.
- [T1] Toda, Y., Curve counting theories via stable objects I. DT/PT correspondence, *J. Amer. Math. Soc.* 23 (2010), no. 4, 11191157.
- [T2] Toda, Y., Curve counting theories via stable objects II. DT/ncDT flop formula,

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, CHINA
E-mail address: kehuazh@mail.sysu.edu.cn